CHAPTER 1 INTRODUCTION

In this Chapter, you will learn:

- types of error,
- ways of error reduction,
- types of software that are used to solve computational problems using numerical methods.

1. ERROR ANALYSIS

Definition of Error

An ***error***, *e* in Numerical Mathematics is the *difference* between the *actual value* (Exact value) and its *computed value*. If x^* represents the computed value of a quantity, the actual value for which is x, then the difference:

$$\begin{aligned} e &= \text{actual value - computed value} \\ &= x - x^* \end{aligned}$$

Ways of Error's Measurement

- **Absolute Error**

$$\begin{aligned} e_{abs} &= |\text{actual value - computed value}| \\ &= |x - x^*| \end{aligned}$$

- **Relative Error**

$$\begin{aligned} e_{rel} &= \frac{e_{abs}}{|x|} \\ &= \frac{|x - x^*|}{|x|} \end{aligned}$$

Types of Error

- **Round-off Error**

Rules for rounding off a number

(a) If a digit to be dropped is 0, 1, 2, 3 or 4: leave the next remaining digit unchanged.

(b) If a digit to be dropped is 5, 6, 7, 8 or 9: increase the next remaining digit by one.

Round-off error is an error to introduce by rounding off numbers to limited number decimal places.

- **Chopping Error**

Number x is chopped to n digits when all digits that follow the n-th digit are discarded and none of the remaining n digits is changed.

- **Truncation Error**

Truncation error is defined as the replacement of one series by another with fewer terms. The error arising from this approximation is called the *truncation error*.
Example: The infinite Taylor Series

$$\exp(x^2) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots + \frac{x^{2n}}{n!} + \ldots$$

might be replaced with the first 5 terms: $\exp(x^2) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}$

Example 1: Given an actual value, x = 1.485642 and its computed value, x^* = 1.492101. Find the following.

(a) Absolute error

$$\begin{aligned} e_{abs} &= |x - x^*| \\ &= |1.485642 - 1.492101| \\ &= |-0.006459| \\ &= 0.006459 \end{aligned}$$

(b) Relative error

$$\begin{aligned} e_{rel} &= \frac{e_{abs}}{|x|} \\ &= \frac{0.006459}{|1.485642|} \\ &= 0.00434762 \end{aligned}$$

Example 2: Round-off the actual value, x = 22/7 to five decimal places and find its absolute error.

$$x = \frac{22}{7} = 3.142857143\ldots \qquad e_{abs} = |x - x^*|$$

$$x^* = \underline{\hspace{4cm}} \qquad =$$

Example 3: Chop the actual value, x = 22/7 to five decimal places and find its relative error.

$$x = \frac{22}{7} = 3.142857143\ldots \qquad e_{rel} = \frac{|x - x^*|}{|x|}$$

$$x^* = \underline{\hspace{4cm}} \qquad =$$

2. ERROR REDUCTION

Nested Form

A polynomial function is given in (2.1):

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n \qquad (2.1)$$

For polynomial evaluation, the rearrangement of terms into *nested form* will sometimes produce a better result. In *nested form*, each power of x is factored out as far as it will go. The *nested form* of a polynomial function is given in (2.2).

$$\begin{aligned} p_n(x) &= a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n \\ &= a_0 + x(a_1 + x(a_2 + \ldots + x(a_{n-1} + a_n x)\ldots)) \end{aligned} \qquad (2.2)$$

Example 4: Consider the following polynomial function with the given value *x*.

$$f(x) = x^3 - 6.1x^2 + 3.2x + 1.5; \; x = 4.71$$

(a) Rewrite $f(x)$ in the nested form.

$$\begin{aligned} f(x) &= x^3 - 6.1x^2 + 3.2x + 1.5 \\ &= x(3.2 + x(x - 6.1)) + 1.5 \end{aligned}$$

(b) Find $f(x)$ in (a) according to three different evaluations below.

(i) Exact evaluation

$$polynomial\ function,\ f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$$

$$\begin{aligned} f(4.71) &= (4.71)^3 - 6.1(4.71)^2 + 3.2(4.71) + 1.5 \\ &= 104.487111 - 6.1(22.1841) + 15.072 + 1.5 \\ &= 104.487111 - 135.32301 + 15.072 + 1.5 \\ &= -14.263899 \end{aligned}$$

$$nested\ form,\ f(x) = x(3.2 + x(x - 6.1)) + 1.5$$

$$\begin{aligned} f(4.71) &= 4.71(3.2 + 4.71(4.71 - 6.1)) + 1.5 & &= 4.71(-3.3469) + 1.5 \\ &= 4.71(3.2 + 4.71(-1.39)) + 1.5 & &= -15.763899 + 1.5 \\ &= 4.71(3.2 - 6.5469) + 1.5 & &= -14.263899 \end{aligned}$$

(ii) Three-digit rounding-off evaluation

$$polynomial\ function,\ f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$$

$$\begin{aligned} f(4.71) &= (4.71)^3 - 6.1(4.71)^2 + 3.2(4.71) + 1.5 \\ &= 104 - 6.1(22.2) + 15.1 + 1.5 \\ &= 104 - 135 + 15.1 + 1.5 \\ &= -14.4 \end{aligned}$$

$$nested\ form,\ f(x) = x(3.2 + x(x - 6.1)) + 1.5$$

$$\begin{aligned} f(4.71) &= 4.71(3.2 + 4.71(4.71 - 6.1)) + 1.5 & &= 4.71(-3.35) + 1.5 \\ &= 4.71(3.2 + 4.71(-1.39)) + 1.5 & &= -15.8 + 1.5 \\ &= 4.71(3.2 - 6.55) + 1.5 & &= -14.3 \end{aligned}$$

(iii) Three-digit chopping evaluation

$$\textit{polynomial function},\ f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$$

$$\begin{aligned} f(4.71) &= (4.71)^3 - 6.1(4.71)^2 + 3.2(4.71) + 1.5 \\ &= 104 - 6.1(22.1) + 15 + 1.5 \\ &= 104 - 134 + 15 + 1.5 \\ &= -13.5 \end{aligned}$$

$$\textit{nested form},\ f(x) = x(3.2 + x(x - 6.1)) + 1.5$$

$$\begin{aligned} f(4.71) &= 4.71(3.2 + 4.71(4.71 - 6.1)) + 1.5 &&= 4.71(-3.34) + 1.5 \\ &= 4.71(3.2 + 4.71(-1.39)) + 1.5 &&= -15.7 + 1.5 \\ &= 4.71(3.2 - 6.54) + 1.5 &&= -14.2 \end{aligned}$$

(c) Find the relative error for the following

Polynomial Function

(i) Exact evaluation and the three-digit rounding-off evaluation

$$\textit{actual value},\ x = f(4.71) = -14.263899$$
$$\textit{computed value},\ x^* = f(4.71) = -14.4$$
$$e_{rel} = \frac{|x - x^*|}{|x|} = \frac{|-14.263899 + 14.4|}{|-14.263899|} = \frac{0.136101}{14.263899} = 0.0095416$$

(ii)Exact evaluation and the three-digit chopping evaluation

$$\textit{actual value},\ x = f(4.71) = -14.263899$$
$$\textit{computed value},\ x^* = f(4.71) = -13.5$$
$$e_{rel} = \frac{|x - x^*|}{|x|} = \frac{|-14.263899 + 13.5|}{|-14.263899|} = \frac{|-0.763899|}{14.263899} = \frac{0.763899}{14.263899} = 0.05355$$

Nested Form

(i) Exact evaluation and the three-digit rounding-off evaluation

$$\textit{actual value},\ x = f(4.71) = -14.263899$$
$$\textit{computed value},\ x^* = f(4.71) = -14.3$$
$$e_{rel} = \frac{|x - x^*|}{|x|} = \frac{|-14.263899 + 14.3|}{|-14.263899|} = \frac{0.036101}{14.263899} = 0.002531$$

(ii)Exact evaluation and the three-digit chopping evaluation

$$\begin{aligned}
&actual\,value,\, x = f(4.71) = -14.263899 \\
&computed\,value,\, x^* = f(4.71) = -14.2 \\
&e_{rel} = \frac{|x - x^*|}{|x|} = \frac{|-14.263899 + 14.2|}{|-14.263899|} = \frac{|-0.063899|}{14.263899} = \frac{0.063899}{14.263899}\, 0.00448
\end{aligned}$$

**Accuracy loss due to the round-off and chopping errors can be reduced by rearranging the polynomial function into nested form.*

<u>Avoiding Loss of Significance in Subtraction</u>

Loss of significance occurs when nearly equal numbers are subtracted. Consider two numbers which are nearly equal, $p = 0.31415926\,536$ and $q = 0.31415957\,341$.

$$\begin{aligned}
p - q &= 0.31415926\,536 - 0.31415957\,341 \\
&= -0.00000308\,05
\end{aligned}$$

After subtraction, their difference is $-0.00000308\,05$ with only FIVE decimal digits of significance. This phenomenon is called loss of significance. There are various techniques that can be used to avoid loss of significance such as the use of rationalization and Taylor series.

- **Rationalization**

Rationalizing is removing the radical in the numerator or denominator:

$$\begin{aligned}
f(x) &= \sqrt{x+4} - 2 \\
&= (\sqrt{x+4} - 2)\frac{(\sqrt{x+4} + 2)}{(\sqrt{x+4} + 2)} \\
&= \frac{x}{(\sqrt{x+4} + 2)}
\end{aligned}$$

As as result, this prodecure allows original terms to be cancelled off and thereby removes the subtraction.

Example 5: Consider the following function:

$$f(x) = x(\sqrt{x+1} - \sqrt{x})$$

i) Approximation of $f(500)$ for given $f(x)$ correctly to six-digit and rounding:

$$f(500) = 500(\sqrt{501} - \sqrt{500})$$
$$= 500(22.3830 - 22.3607)$$
$$=11.1500$$

ii) Rewrite the given function in a way that avoids the loss of significance:

$$f(x) = x(\sqrt{x+1} - \sqrt{x}) * \frac{(\sqrt{x+1} + \sqrt{x})}{(\sqrt{x+1} + \sqrt{x})}$$
$$= \frac{x}{\sqrt{x+1} + \sqrt{x})}$$

iii) Approximation of $f(500)$ for $f(x)$ from (ii) correctly to six-digit and rounding:

$$f(500) = \frac{500}{\sqrt{501} + \sqrt{500})}$$
$$=11.1748$$

iv) Compare the results of (i) and (iii). The actual value is 11.1748.
For (i), $e_{abs} = |x - x^*| = |11.1748 - 11.1500| = 0.0248$
For (ii), $e_{abs} = |x - x^*| = |11.1748 - 11.1748| = 0$

***Accuracy loss due to the round-off can be reduced by avoiding loss of significant digits.*

- **The use of Taylor Series**

Taylor series can be used to remove the subtraction from the nearly equal numbers' subtraction. The terms can be converted into the Taylor series. The Taylor series expansion for some Trigonometry functions is given in the following.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

Example 6: Consider the following function:

$$f(x) = \tan x - \sin x$$

i) Rewrite the given function to avoid loss of significance by using first two nonzero terms in the Taylor series expansion:

$$f(x) = (x + \frac{x^3}{3}) - (x - \frac{x^3}{3!})$$

$$= \frac{1}{2}x^3$$

ii) Approximation of $f(0.0125)$ for $f(x)$ from (i) correctly seven-digit and rounding:

$$f(0.0125) = \frac{1}{2}(0.0125)^3$$

$$= 10^{-6}$$

iii) Find the absolute error given that actual value is $9.766x10^{-7}$.

3. INTRODUCTION TO SOFTWARE

Programming languages such as C, C++ and Java can be used to solve the numerical or mathematical problems. However, interactive computing environments such as FreeMat, Octave and MATLAB provide powerful, built-in mathematical capabilities and a very high-level programming language for rapid mathematical problem solving. **FreeMat** is a free open source numerical computing environment and programming language, similar to MATLAB and GNU Octave. FreeMat integrates extensive mathematical capabilities, especially in linear algebra, with powerful scientific visualization, a high-level programming language and a variety of toolboxes.

FreeMat is chosen in this course due to the several reasons as stated below:

i) FreeMat supports many MATLAB functions and some IDL functionalities, it features a codeless interface to external C and C++,

ii) Superior built-in documentation: FreeMat provides a good documentation.

iii) FreeMat is a free environment for rapid numerical computing, engineering and scientific prototyping and data processing.

iv) FreeMat is user friendly for implementing matrix algebra based calculations.

v) Easier to pick up for the beginning users to learn and use the software to solve the numerical problems.